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The Casimir invariants and Gel'fand basis of the graded unitary group SU(m/n)

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Abstract. In the space of f-particle product states, the Casimir invariants of the graded unitary group SU(m/n) are functions of the class operators of the graded coordinate permutation group $\mathring{S}(f)$, which are equal to the class operators of the graded state permutation group $\mathring{\mathscr{S}}(f)$. Formulae for the quadratic and cubic Casimir invariants are given. It is shown that in general the eigenvalue formulae for the U(m/n) Casimir invariants as functions of the partition can be obtained from those for U($m+n$) by simply changing the argument of the functions from ($m+n$) to ($m-n$). The quasi-standard basis of $\mathring{\mathscr{S}}(f)$ is identified with the Gel'fand basis of SU(m/n), i.e. the irreducible basis classified according to the group chain SU(m/n) ⊃ SU($m/n-1$) ⊃ . . . SU(m) ⊃ . . . SU(2) ⊃ U(1). The special Gel'fand basis of SU(m/n) (for which the weight is restricted to (1, 1, . . . , 1)) can be easily obtained from the Gel'fand basis of SU($m+n$) with due allowances for modifying sign factors arising from the grading. The construction of a general SU(m/n) Gel'fand basis is also discussed.

1. Introduction

The graded unitary group SU(m/n) is of interest both in particle physics (Ne'eman 1979, Dondi and Jarvis 1979, 1980, Taylor 1979), where an internal superalgebra is introduced as a gauge symmetry, and in nuclear physics (Iachello 1980, Balantekin *et al* 1981a, b, Sun and Han 1982a, b), where SU(6/4) is used as a supersymmetry group in nuclei. Several examples of the SU(6/4) supersymmetry in nuclei have been found (Iachello 1980, 1982).

The group SU(m/n) has been studied by Dondi and Jarvis (1981), Han *et al* (1981), Sun and Han (1981, 1982a, b), and Balantekin and Bars (1981a, b). However, up to now these studies mainly concentrated on the irreducible representations (irreps), characters, Kronecker product reductions and branching rules. The other important issues such as the Clebsch–Gordan coefficients and isoscalar factors have not yet been touched upon. As we know, the problem of evaluating the Clebsch–Gordan coefficients and various kinds of isoscalar factors of the ordinary unitary group can be conveniently tackled from the permutation group representation theory (Chen *et al* 1978a, b, Chen 1981). The appealing feature of this approach is that the results obtained are independent of the rank of the particular group under consideration. In a series of papers, the present being the first, we will extend this approach to the graded unitary group and show that the relation between the unitary group and the permutation group is

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exactly the same as that between the graded unitary group and the graded permutation group. A brief account of the subject has been published (Chen *et al* 1983).

In § 2 we prove that the Casimir operators of SU(m/n) are functions of the class operators of the graded permutation group. The graded state permutation group $\mathring{\mathscr{S}}(f)$ is introduced in § 3, and the quasi-standard basis of $\mathring{\mathscr{S}}(f)$ is identified with the SU(m/n) Gel'fand basis in § 4. Finally in § 5 we discuss the construction of the Gel'fand basis of SU(m/n).

In this paper we only treat the so-called class I representations of SU(m/n) (Balantekin and Bars 1981a).

2. The Casimir invariants of SU(m/n) and the CSCO-I of graded permutation group

Let

$$\varphi_A = \begin{pmatrix} \chi_a \\ \psi_\alpha \end{pmatrix}, \tag{1}$$

where $A = a$ (Latin indices) $= 1, 2, \ldots, m$ denote the bosonic or commuting wavefunctions with grade $(A) = 0$, and $A = \alpha$ (Greek indices) $= m+1, \ldots, N = m+n$ denote the fermionic or anticommuting wavefunctions with grade $(A) = 1$. The N^2 infinitesimal generators of the graded unitary group U(m/n) are given by

$$E_{AB} = \sum_{a_1=1}^{f} e_{AB}^{(a_1)}, \qquad e_{AB}^{(a_1)} = \xi_A^{(a_1)\dagger}\xi_B^{(a_1)}, \tag{2a, b}$$

where $\xi_A^{(a_1)\dagger}$ ($\xi_B^{(a_1)}$) is the creation (annihilation) operator for the a_1th particle in the state $A(B)$,

$$e_{AB}^{(a_1)}\varphi_C^{(a_2)} = \delta_{a_1a_2}\delta_{BC}\varphi_A^{(a_1)}. \tag{2c}$$

The graded Lie algebra of U(m/n) is defined by (Jarvis and Green 1979)

$$[E_{AB}, E_{CD}]_{-\left[\begin{smallmatrix}AC\\BD\end{smallmatrix}\right]} = \delta_{BC}E_{AD} - \begin{bmatrix} AC \\ BD \end{bmatrix}\delta_{AD}E_{CB}, \tag{3}$$

where $\left[\begin{smallmatrix}AC\\BD\end{smallmatrix}\right]$ is a sign factor. A general definition for the sign factor is

$$\begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_i & B_j \end{bmatrix} = (-1)^{(A_1+\ldots+A_i)(B_1+\ldots+B_j)}, \tag{4}$$

where

$$(A_1 + \ldots + A_i) = (A_1) + \ldots + (A_i)$$

with addition modulo 2. Obviously $[A] = (-1)^{(A)}$.

The infinitesimal generators of SU(m/n) for $m \neq n$ are given by

$$E'_{AB} = E_{AB} - (m-n)^{-1}\delta_{AB}[B]\sum_C E_{CC}. \tag{5}$$

The $N = m+n$ Casimir invariants of U(m/n) are expressed as

$$\begin{aligned} I_1^{m/n} &= \sum_i E_{ii} = f, \\ &\ldots \\ I_k^{m/n} &= \sum_{i_1\ldots i_k} E_{i_1i_2}[i_2]E_{i_2i_3}[i_3]\ldots[i_k]E_{i_ki_1}, \end{aligned} \tag{6}$$

where the indices i run from 1 to N. The $N-1$ Casimir invariants $I_k'^{m/n}$, $k=2,3,\ldots,N$, of SU(m/n) with $m\neq n$ are given by the same expressions with E_{AB} replaced by E'_{AB}. $I_k'^{m/n}$ can be expressed in terms of $I_k^{m/n}$, as

$$I_2'^{m/n}=I_2^{m/n}-(m-n)^{-1}f^2. \tag{7}$$

We note that the following relations are useful for sign calculus:

$$\begin{bmatrix} A & C \\ & D \\ B & E \end{bmatrix}=\begin{bmatrix} A & C \\ B & \end{bmatrix}\begin{bmatrix} AD \\ BE \end{bmatrix}=\begin{bmatrix} & C \\ A & D \\ & E \end{bmatrix}\begin{bmatrix} & C \\ B & D \\ & E \end{bmatrix}, \tag{8}$$

$$\begin{bmatrix} A & B_1 \\ & \vdots \\ A & B_i \end{bmatrix}=1, \qquad [AA]=[A], \qquad [A_1]\ldots[A_i]\begin{bmatrix} A_1 & A_1 \\ \vdots & \vdots \\ A_i & A_i \end{bmatrix}=1.$$

The graded permutation group and super (or graded) Young diagram were introduced by Dondi and Jarvis (1981) and Balantekin and Bars (1981a, b). We reformulate the graded permutation group as follows. Let an f-particle product state be denoted by

$$|A_1A_2\ldots A_f\rangle\equiv\overset{1}{\varphi}_{A_1}\overset{2}{\varphi}_{A_2}\ldots\overset{f}{\varphi}_{A_f}=\xi^{(1)\dagger}_{A_1}\xi^{(2)\dagger}_{A_2}\ldots\xi^{(f)\dagger}_{A_f}|0\rangle, \tag{9}$$

where the superscripts are the coordinate indices. The transposition $(1f)^\circ$ of the graded permutation group $\mathring{S}(f)$ is defined by

$$(1f)^\circ|A_1A_2\ldots A_f\rangle=\overset{f}{\varphi}_{A_1}\overset{2}{\varphi}_{A_2}\ldots\overset{1}{\varphi}_{A_f}$$

$$=\begin{bmatrix} & A_2 \\ A_1 & \vdots \\ & A_f \end{bmatrix}\begin{bmatrix} & A_2 \\ A_f & \vdots \\ & A_{f-1} \end{bmatrix}|A_fA_2\ldots A_{f-1}A_1\rangle. \tag{10}$$

The transposition $(ij)^\circ$ of $\mathring{S}(f)$ can be similarly defined. It is clear that the graded permutation group $\mathring{S}(f)$ is isomorphic to the ordinary permutation group S(f). The space formed by

$$\mathring{p}|A_1A_2\ldots A_f\rangle, \qquad \mathring{p}\in\mathring{S}(f), \tag{11}$$

is referred to as the product state space L.

Using the technique similar to the one used in the ordinary unitary group (Partensky 1972a, b), and the sign calculus formula (8), we can establish the following relations (i)–(iv) in the space L.

(i) $$\sum_{ij}{}_1^N e_{ij}^{(a_k)}[j]e_{ji}^{(a_l)}=(a_ka_l)^\circ. \tag{12}$$

Let us first prove it is true when acting on the state $\varphi^{a_k}_{A_1}\varphi^{a_l}_{A_2}$.

$$\sum_{ij}{}_1^N e_{ij}^{(a_k)}[j]e_{ji}^{(a_l)}(\varphi^{a_k}_{A_1}\varphi^{a_l}_{A_2})$$

$$=\sum_{ij}{}_1^N[j]\begin{bmatrix} A_1 & j \\ & i \end{bmatrix}(e_{ij}^{(a_k)}\varphi^{(a_k)}_{A_1}e_{ji}^{(a_l)}\varphi^{(a_l)}_{A_2})$$

$$=[A_1]\begin{bmatrix} A_1 & A_1 \\ & A_2 \end{bmatrix}(\varphi^{a_k}_{A_2}\varphi^{a_l}_{A_1})=[A_1A_2](\varphi^{a_k}_{A_2}\varphi^{a_l}_{A_1})=(a_ka_l)^\circ\varphi^{a_k}_{A_1}\varphi^{a_l}_{A_2}. \tag{13a}$$

Next we show that (12) is true for any state in L. We may assume $k<l$ without loss of generality.

$$\sum_{ij}^{N} e_{ij}^{(a_k)}[j]e_{ji}^{(a_l)}(\varphi_{A_1}^{a_1}\ldots\varphi_{A_k}^{a_k}\ldots\varphi_{A_l}^{a_l}\ldots\varphi_{A_f}^{a_f})$$

$$=\sum_{ij}^{N}[j]\begin{bmatrix} i & A_1 \\ & \vdots \\ j & A_{k-1}\end{bmatrix}\begin{bmatrix} j & A_1 \\ & \vdots \\ i & A_{l-1}\end{bmatrix}(\varphi_{A_1}^{a_1}\ldots e_{ij}^{(a_k)}\varphi_{A_k}^{a_k}\ldots e_{ji}^{(a_l)}\varphi_{A_l}^{a_l}\ldots\varphi_{A_f}^{a_f})$$

$$=[A_k]\begin{bmatrix} A_l & A_1 \\ & \vdots \\ A_k & A_{k-1}\end{bmatrix}\begin{bmatrix} A_k & A_1 \\ & \vdots \\ A_l & A_{l-1}\end{bmatrix}(\varphi_{A_1}^{a_1}\ldots\varphi_{A_l}^{a_k}\ldots\varphi_{A_k}^{a_l}\ldots\varphi_{A_f}^{a_f})$$

$$=\begin{bmatrix} & A_{k+1} \\ A_k & \vdots \\ & A_{l-1}\end{bmatrix}\begin{bmatrix} & A_k \\ A_l & \vdots \\ & A_{l-1}\end{bmatrix}(\varphi_{A_1}^{a_1}\ldots\varphi_{A_l}^{a_k}\ldots\varphi_{A_k}^{a_l}\ldots\varphi_{A_f}^{a_f})$$

$$=(a_k a_l)^{\circ}(\varphi_{A_1}^{a_1}\ldots\varphi_{A_k}^{a_k}\ldots\varphi_{A_l}^{a_l}\ldots\varphi_{A_f}^{a_f}). \tag{13b}$$

(ii)
$$\sum_{i_1\ldots i_k}^{N} e_{i_1i_2}^{(a_1)}[i_2]e_{i_2i_3}^{(a_2)}\ldots[i_k]e_{i_ki_1}^{(a_k)}=(a_1a_2\ldots a_k)^{\circ}. \tag{14}$$

We first prove it holds for the state

$$\Phi=\varphi_{A_1}^{a_1}\varphi_{A_2}^{a_2}\ldots\varphi_{A_k}^{a_k}.$$

By shifting firstly the operator $e_{i_ki_1}^{(a_k)}$ to the position in front of $\varphi_{A_k}^{a_k}$, and secondly $e_{i_{k-1}i_k}^{(a_{k-1})}$ in front of $\varphi_{A_{k-1}}^{a_{k-1}},\ldots$, we have

$$\sum_{i_1\ldots i_k}^{N} e_{i_1i_2}^{(a_1)}[i_2]e_{i_2i_3}^{(a_2)}\ldots[i_k]e_{i_ki_1}^{(a_k)}\Phi$$

$$=\sum_{i_1\ldots i_k}^{N}[i_2]\ldots[i_k]\begin{bmatrix} i_2 & A_1 \\ i_3 & \end{bmatrix}\ldots\begin{bmatrix} i_k & A_1 \\ & \vdots \\ i_1 & A_{k-1}\end{bmatrix}e_{i_1i_2}^{(a_1)}\varphi_{A_1}^{a_1}e_{i_2i_3}^{(a_2)}\varphi_{A_2}^{a_2}\ldots e_{i_ki_1}^{(a_k)}\varphi_{A_k}^{a_k}$$

$$=[A_1]\ldots[A_{k-1}]\begin{bmatrix} A_1 & A_1 \\ A_2 & \end{bmatrix}$$

$$\times\begin{bmatrix} A_2 & A_1 \\ A_3 & A_2\end{bmatrix}\ldots\begin{bmatrix} A_{k-1} & A_1 \\ & \vdots \\ A_k & A_{k-1}\end{bmatrix}(\varphi_{A_k}^{a_1}\varphi_{A_1}^{a_2}\ldots\varphi_{A_{k-1}}^{a_k})$$

$$=\begin{bmatrix} & A_1 \\ A_k & \vdots \\ & A_{k-1}\end{bmatrix}(\varphi_{A_k}^{a_1}\varphi_{A_1}^{a_2}\ldots\varphi_{A_{k-1}}^{a_k})=(a_1a_2\ldots a_k)^{\circ}\Phi. \tag{15a}$$

Similarly, for a general state

$$\Psi=\varphi_{A_1}^{1}\ldots\varphi_{A_{a_1}}^{a_1}\ldots\varphi_{A_{a_2}}^{a_2}\ldots\varphi_{A_{a_k}}^{a_k}\ldots\varphi_{A_f}^{f}$$

we have

$$\sum_{i_1\ldots i_k}^{N} e_{i_1i_2}^{(a_1)}[i_2]e_{i_2i_3}^{(a_2)}\ldots[i_k]e_{i_ki_1}^{(a_k)}\Psi$$

$$=[Aa_1]\ldots[Aa_{k-1}]\begin{bmatrix} Aa_k & A_1 \\ & \vdots \\ Aa_1 & Aa_1-1\end{bmatrix}\begin{bmatrix} Aa_1 & A_1 \\ & \vdots \\ Aa_2 & Aa_2-1\end{bmatrix}\ldots\begin{bmatrix} Aa_{k-1} & A_1 \\ & \vdots \\ Aa_k & Aa_k-1\end{bmatrix}$$

$$\times(\varphi^{1}_{A_1}\ldots\varphi^{a_1}_{Aa_k}\ldots\varphi^{a_2}_{Aa_1}\ldots\varphi^{a_k}_{Aa_{k-1}}\ldots\varphi^{f}_{A_f})$$

$$=\begin{bmatrix} & Aa_1+1 \\ Aa_1 & \vdots \\ & Aa_2-1\end{bmatrix}\begin{bmatrix} & Aa_2+1 \\ Aa_2 & \vdots \\ & Aa_3-1\end{bmatrix}\ldots\begin{bmatrix} & Aa_{k-1}+1 \\ Aa_{k-1} & \vdots \\ & Aa_k-1\end{bmatrix}\begin{bmatrix} & Aa_1 \\ Aa_k & \vdots \\ & Aa_k-1\end{bmatrix}$$

$$\times(\varphi^{1}_{A_1}\ldots\varphi^{a_1}_{Aa_k}\ldots\varphi^{a_2}_{Aa_1}\ldots\varphi^{a_k}_{Aa_{k-1}}\ldots\varphi^{f}_{A_f})$$

$$=(a_1a_2\ldots a_k)^{\circ}\Psi. \qquad (15b)$$

(iii) From (14) we obtain

$$P_k^{(a_1\ldots a_k)}\equiv\frac{1}{k!}\sum_{i_1\ldots i_k}^{N}(e_{i_1i_2}^{(a_1)}[i_2]e_{i_2i_3}^{(a_2)}\ldots[i_k]e_{i_ki_1}^{(a_k)})_s=\mathring{C}_{(k)}(k)/(k-1)! \qquad (16a)$$

where the subscript s indicates a symmetrisation in the indices $a_1\ldots a_k$, and $\mathring{C}_{(k)}(k)$ is the k-cycle class operator of $\mathring{S}(k)$.

(iv)
$$P_k^f\equiv\binom{f}{k}^{-1}\sum_{a_1<\ldots<a_k}^{f}P_k^{(a_1\ldots a_k)}=\mathring{C}_{(k)}(f)/g_{(k)}, \qquad g_{(k)}=\binom{f}{k}(k-1)!, \qquad (16b)$$

where $\mathring{C}_{(k)}(f)$ is the k-cycle class operator of $\mathring{S}(f)$, $g_{(k)}$ the number of elements in the class, and $C_{(k)}(f)/g_{(k)}$ the so-called average class operator. Equation (16) is a generalisation of equations (V.3) and (V.5) in Partensky (1972b).

Now we turn to the problem of the relation between the Casimir operators of SU(m/n) and $\mathring{S}(f)$. According to Chen *et al* (1977a), or Chen and Gao (1982), the Casimir operators, or the CSCO-I (the complete set of commuting operators of the first kind), of the permutation group S(f) are found to be

$$C=C_{(2)}(f), \qquad \text{for } 2\leqslant f\leqslant 5, f=7,$$
$$C=(C_{(2)}(f), C_{(3)}(f)) \qquad \text{for } f=6, 8\leqslant f\leqslant 14. \qquad (17)$$

Due to the isomorphism between S(f) and $\mathring{S}(f)$, the CSCO-I of $\mathring{S}(f)$ for $f\leqslant 14$ is either $\mathring{C}_{(2)}(f)$ or $(\mathring{C}_{(2)}(f), \mathring{C}_{(3)}(f))$. We will show that the Casimir operators of SU(m/n) are functions of the CSCO-I of $\mathring{S}(f)$. To this end we only need to show that the former are functions of the class operators of $\mathring{S}(f)$, since any class operator of a finite group is a function of the CSCO-I of the group (Zu 1980).

In analogy with the derivation in Partensky (1972b), we can easily obtain the following relations:

$$P_2^f=[I_2^{m/n}-(m-n)f]/[f(f-1)],$$
$$P_3^f=\{I_3^{m/n}-2(m-n)I_2^{m/n}+[(m-n)^2+1]f-f^2\}/[f(f-1)(f-2)]. \qquad (18)$$

With (16) and (18), the Casimir invariants of U(m/n) can be expressed in terms of the class operators of $\mathring{S}(f)$,

$$I_1^{m/n}=\sum_i{}_1^N E_{ii}=f, \qquad I_2^{m/n}=2\mathring{C}_{(2)}(f)+(m-n)f,$$
$$I_3^{m/n}=3\mathring{C}_{(3)}(f)+4(m-n)\mathring{C}_{(2)}(f)+[(m-n)^2-1]f+f^2. \tag{19a}$$

Higher-power Casimir invariants can be obtained similarly. In general we have

$$I_k^{m/n}=F_k^{m/n}(\mathring{C}_{(k)}(f),\ldots,\mathring{C}_{(2)}(f),m-n). \tag{19b}$$

When $n=0$, (19) reduces to the formula given by Partensky for the Casimir invariants of the U(m) group.

The Casimir invariants of SU(m/n) can be found from (5) and (19). Again they are functions of the class operators of $\mathring{S}(f)$ as well as the quantity $(m-n)$:

$$I_k'^{m/n}=\bar{F}_k^{m/n}(\mathring{C}_{(k)}(f),\ldots,\mathring{C}_{(2)}(f),m-n), \tag{20a}$$

$$I_2'^{m/n}=2\mathring{C}_{(2)}(f)+(m-n)f-f^2/(m-n). \tag{20b}$$

Equations (19) and (20) imply that the Casimir invariants of U(m/n) or SU(m/n) are functions of the CSCO-I of $\mathring{S}(f)$. Therefore, if a basis vector belongs to the irrep ν of $\mathring{S}(f)$, it must also belong to the irrep ν of SU(m/n), and *vice versa* (Chen *et al* 1977a, b). Consequently we can use the partition or Young diagram to label irreps of SU(m/n) and $\mathring{S}(f)$. Besides, the eigenvalues of $\mathring{C}_{(k)}(f)$ and $C_{(k)}(f)$ are identical. The eigenvalues of $C_{(k)}(f)$ for $k=2-4$ were given by Partensky (1972a) as functions of the partition $[\nu]=[\nu_1\nu_2\ldots]$. For instance

$$\lambda_{(2)}^{[\nu]}(f)=\tfrac{1}{2}\Big[f+\sum_l \nu_l(\nu_l-2l)\Big],$$
$$\lambda_{(3)}^{[\nu]}(f)=\tfrac{1}{3}\Big[2f-\tfrac{3}{2}f^2+\sum_l(\nu_l^3-3l\nu_l^2+\tfrac{3}{2}\nu_l^2+3l(l-1)\nu_l)\Big]. \tag{21}$$

By replacing $\mathring{C}_{(k)}(f)$ in (19) or (20) by $\lambda_{(k)}^{[\nu]}(f)$, we get the eigenvalues of U(m/n) or SU(m/n) Casimir invariants. For example

$$I_2'^{m/n}=\sum_l \nu_l(\nu_l-2l)+(m-n+1)f-f^2/(m-n). \tag{22}$$

Equation (22) is a generalisation of (5.23) in Balantekin and Bars (1981a), which was obtained from the character approach with much more labour and is applicable only to the totally symmetric case, i.e. $[\nu]=[f]$.

3. The graded state permutation group

The discussion about the ordinary state permutation group (Chen *et al* 1977b, Chen and Gao 1982) can be carried over to the graded state permutation group $\mathring{\mathscr{S}}(f)$ defined below.

3.1. *The case for f particles occupying f distinct single particle* (SP) *states* $A_1, A_2, \ldots, A_f$

We first specify the ordering of the SP states as $A_1<A_2<\ldots<A_f$, and call the product state

$$|\omega\rangle=|A_1A_2\ldots A_f\rangle \tag{23}$$

the normal order state. Similar to (10), we define a graded state permutation $(A_iA_l)^\circ$ by

$$(A_iA_l)^\circ|A_p\ldots A_iA_j\ldots A_kA_l\ldots A_q\rangle = \begin{bmatrix} & A_j \\ A_i & \vdots \\ & A_l \end{bmatrix}\begin{bmatrix} & A_j \\ A_l & \vdots \\ & A_k \end{bmatrix}|A_p\ldots A_lA_j\ldots A_kA_i\ldots A_q\rangle. \tag{24}$$

The $f-1$ transpositions $(A_iA_{i+1})^\circ$, $i=1,2,\ldots,f-1$, generate the graded state permutation group $\mathring{\mathscr{S}}(f)$. Obviously the graded permutation groups in coordinate indices and in state indices are isomorphic and commutative (i.e. any element of $\mathring{\mathrm{S}}(f)$ commutes with any element of $\mathring{\mathscr{S}}(f)$). In analogy with the ordinary case (Chen and Gao 1982), it can be readily shown that the CSCO-I of $\mathring{\mathrm{S}}(f)$ and $\mathring{\mathscr{S}}(f)$ are equal; however, the CSCO-I of the subgroups $\mathring{\mathrm{S}}(f')$ ($\in\mathring{\mathrm{S}}(f)$) and $\mathring{\mathscr{S}}(f')$ ($\in\mathring{\mathscr{S}}(f)$) are not. Let $\mathring{C}(f')$ and $\mathring{\mathscr{C}}(f')$ be the CSCO-I of $\mathring{\mathrm{S}}(f')$ and $\mathring{\mathscr{S}}(f')$, respectively. The set of commuting operators

$$M=(\mathring{C}(f),\mathring{C}(s)),\qquad \mathring{C}(s)=(\mathring{C}(f-1),\ldots,\mathring{C}(2)), \tag{25}$$

is called the CSCO-II of $\mathring{\mathrm{S}}(f)$, while

$$K=(\mathring{C}(f),\mathring{C}(s),\mathring{\mathscr{C}}(s)),\qquad \mathring{\mathscr{C}}(s)=(\mathring{\mathscr{C}}(f-1),\ldots,\mathring{\mathscr{C}}(2)), \tag{26}$$

is called the CSCO-III of $\mathring{\mathrm{S}}(f)$. The eigenvectors of K are the Yamanouchi bases of $\mathring{\mathrm{S}}(f)$ and $\mathring{\mathscr{S}}(f)$, i.e. irreducible bases in the classification scheme of $\mathring{\mathrm{S}}(f)\supset\mathring{\mathrm{S}}(f-1)\supset\ldots\supset\mathring{\mathrm{S}}(2)$ and $\mathring{\mathscr{S}}(f)\supset\mathring{\mathscr{S}}(f-1)\supset\ldots\supset\mathring{\mathscr{S}}(2)$, and can be designated by $|\mathring{Y}_m^{(\nu)},\mathring{W}_k^{(\nu)}\rangle$, where $\mathring{Y}_m^{[\nu]}$ and $\mathring{W}_k^{[\nu]}$ are called the graded Young tableaux and graded Weyl tableaux, respectively. Thus

$$\begin{pmatrix}\mathring{C}(f)\\ \mathring{C}(s)\\ \mathring{\mathscr{C}}(s)\end{pmatrix}|\mathring{Y}_m^{[\nu]},\mathring{W}_k^{[\nu]}\rangle=\begin{pmatrix}\nu\\ m\\ k\end{pmatrix}|\mathring{Y}_m^{[\nu]},\mathring{W}_k^{[\nu]}\rangle, \tag{27}$$

where ν, m and k are eigenvalues of $\mathring{C}(f)$, $\mathring{C}(s)$ and $\mathring{\mathscr{C}}(s)$. There is a one-to-one correspondence between (ν,m) or (ν,k) and the Yamanouchi symbols (Chen and Gao 1982).

For the case without grading, (27) reduces to

$$\begin{pmatrix}C(f)\\ C(s)\\ \mathscr{C}(s)\end{pmatrix}|Y_m^{[\nu]},W_k^{[\nu]}\rangle=\begin{pmatrix}\nu\\ m\\ k\end{pmatrix}|Y_m^{[\nu]},W_k^{[\nu]}\rangle, \tag{28}$$

where $Y_m^{[\nu]}$ and $W_k^{[\nu]}$ are the ordinary Young tableaux and Weyl tableaux, and $|Y_m^{[\nu]},W_k^{[\nu]}\rangle$ are the Yamanouchi bases of $\mathrm{S}(f)$ and $\mathscr{S}(f)$.

It is clear that the transformation of $|\mathring{Y}_m^{[\nu]},\mathring{W}_k^{[\nu]}\rangle$ under the graded permutation group $\mathring{\mathrm{S}}(f)(\mathring{\mathscr{S}}(f))$ is exactly the same as $|Y_m^{[\nu]},W_k^{[\nu]}\rangle$ under the permutation group $\mathrm{S}(f)(\mathscr{S}(f))$. We know that the basis vector $|Y_m^{[\nu]},W_k^{[\nu]}\rangle$ can be expressed as

$$\begin{aligned}|Y_m^{[\nu]},W_k^{[\nu]}\rangle &= P_m^{[\nu]k}|\omega\rangle=\left(\frac{h_\nu}{f!}\right)^{1/2}\sum_p D_{mk}^{[\nu]}(p)p|\omega\rangle\\ &=\sum_{1'\ldots f'} u_{\nu mk,(1'\ldots f')}|A_{1'}A_{2'}\ldots A_{f'}\rangle,\end{aligned} \tag{29}$$

where $P_m^{[\nu]k}$ is the normalised projection operator, $D_{mk}^{[\nu]}$ the Yamanouchi matrix elements, h_ν the dimension of the irrep $[\nu]$ of S(f), and $u_{\nu mk,(1'\ldots f')}$ with $i'=p(i)$ the expansion coefficients. Similarly for the basis vector $|\mathring{Y}^{[\nu]}_m, \mathring{W}^{[\nu]}_k\rangle$ we have

$$|\mathring{Y}^{[\nu]}_m, \mathring{W}^{[\nu]}_k\rangle = \mathring{P}^{[\nu]k}_m|\omega\rangle = \left(\frac{h_\nu}{f!}\right)^{1/2}\sum_p D^{[\nu]}_{mk}(p)\mathring{p}|\omega\rangle$$

$$= \sum_{1'\ldots f'} u_{\nu mk,(1'\ldots f')} \prod_{\substack{i<j\\ A_{i'}>A_{j'}}} [A_{i'}A_{j'}]|A_{1'}A_{2'}\ldots A_{f'}\rangle. \tag{30}$$

Comparing (29) with (30), it is seen that the basis vector $|\mathring{Y}^{[\nu]}_m, \mathring{W}^{[\nu]}_k\rangle$ can be obtained from $|Y^{[\nu]}_m, W^{[\nu]}_k\rangle$ by the substitution

$$|A_{1'}A_{2'}\ldots A_{f'}\rangle \to \prod_{\substack{i<j\\ A_{i'}>A_{j'}}} [A_{i'}A_{j'}]|A_{1'}A_{2'}\ldots A_{f'}\rangle.$$

For example, from table V in Chen and Gao (1982), we have the Yamanouchi bases of $\mathring{S}(3)$ and $\mathring{\mathscr{S}}(3)$ listed in table 1.

For instance

$$\left|\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline A&C\\\hline B\\\cline{1-1}\end{array}\right\rangle = -\tfrac{1}{2}\Big(\left[A\ {}^{B}_{C}\right][BC]|CBA\rangle - [BC]|ACB\rangle$$

$$-\left[C\ {}^{A}_{B}\right]|CAB\rangle + \left[A\ {}^{B}_{C}\right]|BCA\rangle\Big)$$

which is symmetric (antisymmetric) under the graded coordinate (state) permutation $(12)^\circ((AB)^\circ)$.

It also follows from (29) and (30) that if the f SP states are all bosonic, then

$$|\mathring{Y}^\nu_m, \mathring{W}^\nu_k\rangle = |Y^\nu_m, W^\nu_k\rangle, \tag{31a}$$

and if they are all fermionic, then

$$|\mathring{Y}^\nu_m, \mathring{W}^\nu_k\rangle = \begin{cases} \Lambda^\nu_m\Lambda^\nu_k|\tilde{Y}^\nu_m, \tilde{W}^\nu_k\rangle & \text{under Young–Yamanouchi's PC,}\\ |\tilde{Y}^\nu_m, \tilde{W}^\nu_k\rangle & \text{under Jahn's PC,}\end{cases} \tag{31b}$$

where Λ^ν_m are the phase factors (Hamermesh 1962), $\tilde{Y}^\nu_m(\tilde{W}^\nu_k)$ the conjugation (interchange rows with columns) of the Young (Weyl) tableau $Y^\nu_m(W^\nu_k)$, and the upper equation on the right of (31b) refers to the Young–Yamanouchi phase convention which will be used henceforth, while the lower one refers to the Jahn phase convention. For example with the Young–Yamanouchi phase convention

$$\left|\overset{\circ}{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}\ \overset{\circ}{\begin{array}{|c|c|}\hline \alpha&\beta\\\hline \gamma\\\cline{1-1}\end{array}}\right\rangle = \left|\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline \alpha&\gamma\\\hline \beta\\\cline{1-1}\end{array}\right\rangle, \qquad \left|\overset{\circ}{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}}\ \overset{\circ}{\begin{array}{|c|c|}\hline \alpha&\gamma\\\hline \beta\\\cline{1-1}\end{array}}\right\rangle = -\left|\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline \alpha&\beta\\\hline \gamma\\\cline{1-1}\end{array}\right\rangle.$$

3.2. *The case for f particles occupying $N<f$ SP states*

Suppose we have a configuration $(1)^{f_1}(2)^{f_2}\ldots(N)^{f_N}$, with $1, 2, \ldots, m$ referring to boson SP states, and $m+1, \ldots, N$ referring to fermion SP states, $f=\Sigma_{i=1}^N f_i$. The N SP states are assigned to the f state indices $A_1\ldots A_f$ in the following manner:

$$A_1=A_2=\ldots=A_{f_1}=1, \qquad A_{f_1+1}=\ldots=A_{f_1+f_2}=2,$$
$$\ldots A_{f-f_N+1}=\ldots=A_f=N. \tag{32a}$$

Once this is done, we can again use (24) as the definition for the graded state

Table 1. The Yamanouchi bases of $\mathring{S}(3)$ and $\mathring{\mathscr{S}}(3)$.

ν	m	k	$\mathring{Y}_m^{[\nu]}$	$\mathring{W}_k^{[\nu]}$	$\|ABC\rangle$	$[AB]\|BAC\rangle$	$\begin{bmatrix}A & B \\ & C\end{bmatrix}[BC]\|CBA\rangle$	$[BC]\|ACB\rangle$	$\begin{bmatrix}C & A \\ & B\end{bmatrix}\|CAB\rangle$	$\begin{bmatrix}A & B \\ & C\end{bmatrix}\|BCA\rangle$
3	1	1	$\begin{matrix}1&2&3\end{matrix}$	$\begin{matrix}A&B&C\end{matrix}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$
0	1	1	$\begin{matrix}1&2\\3\end{matrix}$	$\begin{matrix}A&B\\C\end{matrix}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$
0	−1	1	$\begin{matrix}1&3\\2\end{matrix}$	$\begin{matrix}A&B\\C\end{matrix}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
0	1	−1	$\begin{matrix}1&2\\3\end{matrix}$	$\begin{matrix}A&C\\B\end{matrix}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
0	−1	−1	$\begin{matrix}1&3\\2\end{matrix}$	$\begin{matrix}A&C\\B\end{matrix}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{12}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$
−3	−1	−1	$\begin{matrix}1\\2\\3\end{matrix}$	$\begin{matrix}A\\B\\C\end{matrix}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$

permutation group $\mathring{\mathscr{S}}(f)$. Again, all the results involving the ordinary unitary group and the state permutation group (Chen *et al* 1977b, Chen and Gao 1982) can be transferred to the graded unitary group $SU(m/n)$ and the graded state permutation group $\mathring{\mathscr{S}}(f)$. These results are summarised as follows.

(a). Whenever there are repeated SP states in the product state, a single permutation operator of $\mathring{\mathscr{S}}(f)$ ceases to have a definite meaning; only the class operators of the following subgroups

$$\mathring{\mathscr{S}}(f)\supset\mathring{\mathscr{S}}(f-f_N)\supset\mathring{\mathscr{S}}(f-f_N-f_{N-1})\supset\ldots\supset\mathring{\mathscr{S}}(f_1+f_2)\supset\mathring{\mathscr{S}}(f_1)$$

remain meaningful, namely the result of applying these class operators on a product state is the same irrespective of which one among the f_1 identical SP states, '1' being regarded as A_1, or $A_2,\ldots$ or A_{f_1}, and which one among the f_2 identical SP states, '2' being regarded as $A_{f_1+1},\ldots$ or $A_{f_1+f_2}$, etc. Let

$$\mathring{\mathscr{S}}(s')=(\mathring{\mathscr{S}}(f-f_N)\supset\ldots\supset\mathring{\mathscr{S}}(f_1+f_2)\supset\mathring{\mathscr{S}}(f_1)) \tag{32b}$$

and we refer to $\mathring{\mathscr{S}}(f)\supset\mathring{\mathscr{S}}(s')$ as the broken chain of $\mathring{\mathscr{S}}(f)$. Therefore the CSCO

$$(\mathring{\mathscr{C}}(f);\mathring{\mathscr{C}}(s'))=(\mathring{\mathscr{C}}(f);\mathring{\mathscr{C}}(f-f_N),\ldots,\mathring{\mathscr{C}}(f_1+f_2),\mathring{\mathscr{C}}(f_1)) \tag{33}$$

of the broken chain have a definite meaning, $\mathring{\mathscr{C}}(i)$ being the CSCO-I of $\mathring{\mathscr{S}}(i)$.

(b). The simultaneous eigenvectors of $(\mathring{\mathscr{C}}(f);\mathring{\mathscr{C}}(s'))$ are called the quasi-standard basis of $\mathring{\mathscr{S}}(f)$, and can be labelled by the graded Weyl tableaux $\mathring{W}^{\nu}_{\kappa}$

$$\begin{pmatrix}\mathring{\mathscr{C}}(f)\\ \mathring{\mathscr{C}}(s')\end{pmatrix}|\mathring{W}^{\nu}_{\kappa}\rangle=\begin{pmatrix}\nu\\ \kappa\end{pmatrix}|\mathring{W}^{\nu}_{\kappa}\rangle. \tag{34}$$

The simultaneous eigenvectors of $\mathring{C}(f)\ (=\mathring{\mathscr{C}}(f))$, $\mathring{C}(s)$ and $\mathring{\mathscr{C}}(s')$ are the Yamanouchi basis of $\mathring{S}(f)$ and quasi-standard basis of $\mathring{\mathscr{S}}(f)$:

$$\begin{pmatrix}\mathring{C}(f)\\ \mathring{C}(s)\\ \mathring{\mathscr{C}}(s')\end{pmatrix}|\mathring{Y}^{\nu}_{m},\mathring{W}^{\nu}_{\kappa}\rangle=\begin{pmatrix}\nu\\ m\\ \kappa\end{pmatrix}|\mathring{Y}^{\nu}_{m},\mathring{W}^{\nu}_{\kappa}\rangle. \tag{35}$$

(c). In analogy with (30), $|\mathring{Y}^{\nu}_{m},\mathring{W}^{\nu}_{\kappa}\rangle$ can be expressed in terms of the projection operator acting on the normal order state:

$$|\mathring{Y}^{\nu}_{m},\mathring{W}^{\nu}_{\kappa}\rangle=[\mathring{R}^{[\nu]k}(\omega)]^{-1}\mathring{P}^{[\nu]k}_{m}|\omega\rangle=[\mathring{R}^{[\nu]k}(\omega)]^{-1}|\mathring{Y}^{\nu}_{m}\mathring{W}^{\nu}_{k}\rangle_{\text{ass}} \tag{36}$$

where $\mathring{R}^{[\nu]k}(\omega)$ is a renormalisation constant, and the subscript 'ass' means assimilation (i.e. letting some SP states in the Yamanouchi basis of $\mathring{\mathscr{S}}(f)$ be equal). We note that κ is the set of eigenvalues of the operators $\mathring{\mathscr{C}}(f-f_N),\ldots,\mathring{\mathscr{C}}(f_1+f_2),\mathring{\mathscr{C}}(f_1)$, while k is that of the operators $\mathring{\mathscr{C}}(f-1),\mathring{\mathscr{C}}(f-2),\ldots,\mathring{\mathscr{C}}(2)$. κ can be obtained from k by deleting the eigenvalues of the meaningless operators in the set $\mathring{\mathscr{C}}(s)$. Therefore it may be that several k correspond to the same κ. The absolute value of the norm $\mathring{R}^{[\nu]k}(\omega)$ can be evaluated by the following formula:

$$|\mathring{R}^{[\nu]k}(\omega)|=(\langle\omega|(\mathring{P}^{[\nu]k}_{m})^{\dagger}\mathring{P}^{[\nu]k}_{m}|\omega\rangle)^{1/2}=[(f!/h_{\nu})^{1/2}\langle\omega|\mathring{P}^{[\nu]k}_{k}|\omega\rangle]^{1/2}$$
$$=\Big(\langle\omega|\sum_{p}{}'D^{[\nu]}_{kk}(p)\mathring{p}|\omega\rangle\Big)^{1/2}. \tag{37}$$

The prime means that the summation is restricted to those p which satisfy $\mathring{p}|\omega\rangle=\pm|\omega\rangle$.

Equation (36) shows that the quasi-standard basis of $\mathring{\mathcal{S}}(f)$ can be obtained from the Yamanouchi basis of $\mathring{\mathcal{S}}(f)$ through assimilation and renormalisation.

Let us now address the question of the sign of the norm $\mathring{R}^{[\nu]k}(\omega)$. For clarity, we will ignore the Young tableau and focus our attention on the Weyl tableau. Equation (36) reads

$$\mathring{P}^{[\nu]k}|\omega\rangle = \mathring{R}^{[\nu]k}(\omega)|\mathring{W}^{\nu}_{\kappa}\rangle. \tag{38}$$

Suppose that in the normal order state $|\omega\rangle$, the particles i and $i+1$ occupy the same SP state A_i. Consequently

$$(i, i+1)^{\circ}|\omega\rangle = [A_i]|\omega\rangle. \tag{39}$$

It is readily seen that if the Young tableaux Y^{ν}_{k} and $Y^{\nu}_{k'}$ are related by an interchange of the positions of i and $i+1$,

$$Y^{\nu}_{k'} = (i, i+1)Y^{\nu}_{k}, \tag{40}$$

then k and k' will correspond to the same κ. Using the property of the projection operator (Chen and Gao 1982)

$$\mathring{P}^{[\nu]k}_{m}\mathring{p} = \sum_{k'} D^{[\nu]}_{kk'}(p)\mathring{P}^{[\nu]k'}_{m} \tag{41a}$$

we get

$$\mathring{P}^{[\nu]k}|\omega\rangle = \mathring{P}^{[\nu]k}[A_i](i, i+1)^{\circ}|\omega\rangle = \sum_{k'}[A_i]D^{[\nu]}_{kk'}(i, i+1)\mathring{P}^{[\nu]k'}|\omega\rangle. \tag{41b}$$

Therefore

$$\mathring{P}^{[\nu]k}|\omega\rangle = [\mathring{R}^{[\nu]k}(\omega)/\mathring{R}^{[\nu]k'}(\omega)]\mathring{P}^{[\nu]k'}|\omega\rangle, \tag{42a}$$

$$\mathring{R}^{[\nu]k}(\omega)/\mathring{R}^{[\nu]k'}(\omega) = [A_i]D^{[\nu]}_{kk'}(i, i+1)/\{1-[A_i]D^{[\nu]}_{kk}(i, i+1)\}. \tag{42b}$$

The denominator in (42b) is always positive, and the off-diagonal matrix element $D^{[\nu]}_{kk'}(i, i+1)$ is also positive under the Young–Yamanouchi phase convention. The relative sign between $\mathring{R}^{[\nu]k}(\omega)$ and $\mathring{R}^{[\nu]k'}(\omega)$ is thus decided by $[A_i]$. For the case without grading, the norms $R^{[\nu]k}(\omega)$ are all positive.

The above consideration can be extended to the more general case. For instance, if in $|\omega\rangle$ we have $A_{i-1} = A_i = A_{i+1}$, and $Y_{k''} = (i-1, i)Y_{k'}$, $Y_{k'} = (i, i+1)Y_k$, then the quantum numbers k, k' and k'' correspond to the same κ. We can use (42b) to decide the relative sign between $\mathring{R}^{[\nu]k''}(\omega)$ and $\mathring{R}^{[\nu]k'}(\omega)$, as well as that between $\mathring{R}^{[\nu]k'}(\omega)$ and $\mathring{R}^{[\nu]k}(\omega)$.

Actually the sign of $\mathring{R}^{[\nu]k}(\omega)$ can be fixed by the following simple procedure. We choose the norm $\mathring{R}^{[\nu]k}(\omega)$ with the maximum possible Yamanouchi number k compatible with the quantum number κ being positive. Suppose k and k' correspond to the same κ, and $Y_{k'} = qY_k$. q is necessarily of the form: $q = q'q''$, $q' \in S(f_1)\otimes\ldots\otimes S(f_m)$, $q'' \in S(f_{m+1})\otimes\ldots\otimes S(f_N)$. Then the sign of $\mathring{R}^{[\nu]k}(\omega)$ is given by $\delta_{q''}$, i.e.

$$\mathring{R}^{[\nu]k'}(\omega) = \delta_{q''}\left(\langle\omega|\sum_{p}{}' D^{[\nu]}_{k'k'}(p)\mathring{p}|\omega\rangle\right)^{1/2} \tag{43}$$

where $\delta_{q''}$ is the parity of the permutation q''.

When all SP states are bosonic or fermionic, the norm becomes

$$\mathring{R}^{[\nu]k'}(\omega)=\begin{cases} R^{[\nu]k'}(\omega)=\left(\langle\omega|\sum_p{}' D^{\nu}_{k'k'}(p)p|\omega\rangle\right)^{1/2}, & \text{for bosons,}\\ \delta_q R^{[\tilde{\nu}]\tilde{k}'}(\omega) & \text{for fermions,}\end{cases} \qquad (44a)$$

where $R^{[\nu]k'}(\omega)$ is the norm for the ordinary permutation group.

From (43) it is seen that $\mathring{R}^{[\nu]k}(\omega)$ has the property that when $i_n \neq i_{n+1} \neq \ldots \neq i_m$,

$$\mathring{R}^{[\nu]k}(i_1 i_2 \ldots i_n i_{n+1} \ldots i_m)=\mathring{R}^{[\nu']k'}(i_1 i_2 \ldots i_n), \qquad (44b)$$

where the quantum numbers $[\nu']$ and k' are determined in the following way. If $\mathring{R}^{[\nu]k}(\omega)$ is regarded as a function of the graded Young tableau $\mathring{Y}^{\nu}_k$ and the Weyl tableau $\mathring{W}^{\nu}_{\kappa}$ of (38), i.e.

$$\mathring{R}^{[\nu]k}(\omega)=R(\mathring{Y}^{\nu}_k, \mathring{W}^{\nu}_{\kappa}),$$

then $[\nu']$ is decided by dropping the boxes filled with the indices $i_{n+1} \ldots i_m$ in the tableau $\mathring{W}^{\nu}_{\kappa}$, while k' is decided by dropping the corresponding boxes in the tableau Y^{ν}_k. For example

$$\mathring{R}\begin{pmatrix}12456 & ABCDE\\ 3 & B\end{pmatrix}=\mathring{R}\begin{pmatrix}12 & AB\\ 3 & B\end{pmatrix}.$$

Some examples of the norm are given in table 2.

According to tables 1 and 2, as well as (36), we obtain the quasi-standard bases for a three-particle system shown in table 3.

From (31) it is clear that the rules for writing a graded Young tableau and the ordinary one are the same, while those for a graded Weyl tableau and the ordinary Weyl tableau differ only in one respect, namely for a graded Weyl tableau, besides the requirement that no two identical boson states are allowed to occupy the same column, no two fermion states are allowed to occupy the same row, of a Young diagram. For instance

$$\left|\,\mathring{\begin{matrix}a & B\\ a & \end{matrix}}\,\right\rangle=0 \qquad \left|\,\mathring{\begin{matrix}\alpha & \alpha\\ B & \end{matrix}}\,\right\rangle=0 \qquad \text{but}\left|\,\mathring{\begin{matrix}\alpha & B\\ \alpha & \end{matrix}}\,\right\rangle\neq 0.$$

4. The Gel'fand basis of SU(m/n)

Dondi and Jarvis (1981) and Balantekin and Bars (1981a, b) used the extension of the Young operator technique to construct the irreducible basis of SU(m/n). The disadvantage of the basis thus obtained is that it is not an orthogonal basis. We can extend the definition for the Gel'fand basis of SU(m) to SU(m/n); namely, a Gel'fand basis of SU(m/n) is defined as the basis which is a simultaneous eigenvector of the Casimir invariants of the groups $SU(m/n)\supset SU(m/n-1)\supset\ldots SU(m)\supset\ldots SU(2)\supset U(1)$. This basis is obviously orthogonal.

It is known that the Gel'fand basis of SU(m) can be constructed by applying the projection operator of the permutation group to the normal order state (Lezuo 1972, Patterson and Harter 1976, Sarma and Sahasrabudhe 1980). On the other hand, it

Table 2. The norm $\mathring{R}^{[\nu]k}(\omega)$ for some simple cases.

$[\nu]k$ \ $\|\omega\rangle$	$\|AABC\rangle$	$\|ABBC\rangle$	$\|ABCC\rangle$	$\|AABB\rangle$	$\|AAAB\rangle$	$\|ABBB\rangle$
1 2 3 4	$(1+[A])^{1/2}$	$(1+[B])^{1/2}$	$(1+[C])^{1/2}$	$(1+[A]+[B]+[A][B])^{1/2}$	$(3+3[A])^{1/2}$	$(3+3[B])^{1/2}$
1 2 3 4	$(1+[A])^{1/2}$	$(1+[B])^{1/2}$	$(1-\frac{1}{3}[C])^{1/2}$	$(1+[A]-\frac{1}{3}[B]-\frac{1}{3}[A][B])^{1/2}$	$(3+3[A])^{1/2}$	$(\frac{1}{3}+\frac{1}{3}[B])^{1/2}$
1 2 4 3	$(1+[A])^{1/2}$	$(1-\frac{1}{2}[B])^{1/2}$	$[C](1+\frac{1}{3}[C])^{1/2}$	$[B](1+[A]+\frac{1}{3}[B]+\frac{1}{3}[A][B])^{1/2}$	0	$[B](\frac{2}{3}+\frac{2}{3}[B])^{1/2}$
1 3 4 2	$(1-[A])^{1/2}$	$[B](1+\frac{1}{2}[B])^{1/2}$	$(1+[C])^{1/2}$	$(1-[A]+[B]-[A][B])^{1/2}$	0	$(2+2[B])^{1/2}$
1 2 3 4	$(1+[A])^{1/2}$	$(1+\frac{1}{2}[B])^{1/2}$	$(1+[C])^{1/2}$	$(1+[A]+[B]+[A][B])^{1/2}$	0	0
1 3 2 4	$(1-[A])^{1/2}$	$[B](1-\frac{1}{2}[B])^{1/2}$	$(1-[C])^{1/2}$	$(1-[A]-[B]+[A][B])^{1/2}$	0	0
1 2 3 4	$(1+[A])^{1/2}$	$(1-\frac{1}{2}[B])^{1/2}$	$(1-[C])^{1/2}$	$(1+[A]-[B]-[A][B])^{1/2}$	0	$(2-2[B])^{1/2}$
1 3 2 4	$(1-[A])^{1/2}$	$[B](1+\frac{1}{2}[B])^{1/2}$	$(1-\frac{1}{3}[C])^{1/2}$	$(1-[A]-\frac{1}{3}[B]+\frac{1}{3}[A][B])^{1/2}$	0	$[B](\frac{2}{3}-\frac{2}{3}[B])^{1/2}$
1 4 2 3	$(1-[A])^{1/2}$	$(1-[B])^{1/2}$	$[C](1+\frac{1}{3}[C])^{1/2}$	$[B](1-[A]+\frac{1}{3}[B]-\frac{1}{3}[A][B])^{1/2}$	$(3-3[A])^{1/2}$	$(\frac{1}{3}-\frac{1}{3}[B])^{1/2}$
1 2 3 4	$(1-[A])^{1/2}$	$(1-[B])^{1/2}$	$(1-[C])^{1/2}$	$(1-[A]-[B]+[A][B])^{1/2}$	$(3-3[A])^{1/2}$	$(3-3[B])^{1/2}$

Table 3. The Yamanouchi basis of $\mathring{S}(3)$ and the quasi-standard basis of $\mathscr{S}(3)$.

$\mathring{Y}^{\nu}_{m}$	$\mathring{W}^{\nu}_{\kappa}$	$\mathcal{N}$	$\lvert AAB\rangle$	$[AB]\lvert ABA\rangle$	$\lvert BAA\rangle$
[1 2 3]	[A A B]	$\sqrt{\frac{1}{3}}\{A\}_+$	1	1	1
[1 2 / 3]	[A A / B]	$\sqrt{\frac{1}{6}}\{A\}_+$	2	−1	−1
[1 3 / 2]	[A A / B]	$\sqrt{\frac{1}{2}}\{A\}_+$	0	1	−1
[1 2 / 3]	[A B / A]	$\sqrt{\frac{1}{2}}\{A\}_-$	0	1	1
[1 3 / 2]	[A B / A]	$\sqrt{\frac{1}{6}}\{A\}_-$	2	1	−1
[1 / 2 / 3]	[A / A / B]	$\sqrt{\frac{1}{3}}\{A\}_-$	1	−1	1
$\mathring{Y}^{\nu}_{m}$	$\mathring{W}^{\nu}_{\kappa}$	$\mathcal{N}$	$\lvert ABB\rangle$	$[AB]\lvert BAB\rangle$	$\lvert BBA\rangle$
[1 2 3]	[A B B]	$\sqrt{\frac{1}{3}}\{B\}_+$	1	1	1
[1 2 / 3]	[A B / B]	$\left(\frac{1}{4+2[B]}\right)^{1/2}$	1	1	$-(1+[B])$
[1 3 / 2]	[A B / B]	$[B]\left(\frac{1}{4-2[B]}\right)^{1/2}$	1	−1	$-(1-[B])$
[1 / 2 / 3]	[A / B / B]	$\sqrt{\frac{1}{3}}\{B\}_-$	1	−1	1

$\mathcal{N}$ is the total normalisation constant. $\{A\}_{\pm}=\frac{1}{2}(1\pm[A])$.

has been proved that the quasi-standard basis of the permutation group is just the Gel'fand basis of the unitary group (Chen *et al* 1977b). Now we are going to show that the quasi-standard basis of the graded state permutation group $\mathscr{S}(f)$ is the Gel'fand basis of the graded unitary group SU(m/n). The extension from the ordinary case to the graded one is straightforward. So we skip the proof and simply quote the results.

(a) In analogy with (16*b*), we first define the operator

$$P_k^{(f-f_N)}=\binom{f-f_N}{k}^{-1}\frac{1}{k\,!}\sum_{a_1<\ldots<a_k}^{f}{}_{1}\;\sum_{i_1\ldots i_k}^{N-1}{}_{1}\,(e^{(a_1)}_{i_1i_2}[i_2]e^{(a_2)}_{i_2i_3}\ldots[i_k]e^{(a_k)}_{i_ki_1})_{\mathrm{s}}. \quad (45)$$

Acting on any product state $|\omega_i\rangle=p_i|\omega\rangle$, $p_i\in\mathrm{S}(f)$, the operator $P_k^{(f-f_N)}$ is equivalent to the average k-cycle class operator of the graded state permutation group $\mathscr{S}(f-f_N)$

$$P_k^{(f-f_N)}|\omega_i\rangle=(g'_{(k)})^{-1}\mathring{\mathscr{C}}_{(k)}(f-f_N)|\omega_i\rangle, \qquad g'_{(k)}=\binom{f-f_N}{k}(k-1)\,!. \quad (46)$$

(b) The Casimir operators $I_k^{\prime m/n-1}$ of SU$(m/n-1)$ are functions of the CSCO-I of $\mathscr{S}(f-f_N)$:

$$I_k^{\prime m/n-1}=\bar{F}_k^{m/n-1}(\mathring{\mathscr{C}}(f-f_N)). \quad (47a)$$

Similarly we have

$$
\begin{aligned}
I_k'^{m/n-2} &= \bar{F}_k^{m/n-2}(\mathring{\mathscr{C}}(f - f_N - f_{N-1})), \\
&\cdots \\
I_2'^{(2)} &= \bar{F}_2^{(2)}(\mathring{\mathscr{C}}(f_1 + f_2)).
\end{aligned} \tag{47b}
$$

On the basis of (34), (47) and (20), we know that the quasi-standard basis of $\mathring{\mathscr{S}}(f)$ can be identified with the Gel'fand basis of SU(m/n). Therefore we can use the graded Weyl tableau to label the SU(m/n) Gel'fand basis, and the dimension of an irrep $[\nu]$ of SU(m/n) is equal to the total number of the possible graded Weyl tableaux belonging to the same Young diagram $[\nu]$. A graded Weyl tableau results from filling a Young diagram with $N = m + n$ SP states subject to the following rules. (1) The state indices must be in non-decreasing order from left to right in each row and from top to bottom in each column. (2) No two bosonic (fermionic) states are permitted to sit in the same column (row). For example, the irrep [21] of the SU(2/3) group is of dimension 40 with the following graded Weyl tableaux.

3 bosons:	aa	ab										
	b	b										
2 bosons	aa	aa	aa	ab	$a\alpha$	ab	$a\beta$	ab	$a\gamma$	bb	bb	bb
1 fermion:	α	β	γ	α	b	β	b	γ	b	α	β	γ
1 boson	$a\alpha$	$a\alpha$	$a\beta$	$a\alpha$	$a\gamma$	$a\beta$	$a\beta$	$a\gamma$	$a\gamma$			
2 fermions:	α	β	α	γ	α	β	γ	β	γ			
	$b\alpha$	$b\alpha$	$b\beta$	$b\alpha$	$b\gamma$	$b\beta$	$b\beta$	$b\gamma$	$b\gamma$			
	α	β	α	γ	α	β	γ	β	γ			
3 fermions:	$\alpha\beta$	$\alpha\gamma$	$\alpha\beta$	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\gamma$	$\beta\gamma$				
	α	α	β	γ	β	β	γ	γ				

The explicit expressions for the above SU(2/3) Gel'fand basis vectors can be read out from table 1 or table 3.

5. Summary

The study presented in this paper shows that it is very easy to treat the representation problem U(m/n) in terms of the graded permutation group. With some slight modifications, we can take over all the results obtained in the representation theory of the unitary and permutation groups. In table 4 we indicate the necessary modifications for transferring the results concerning the groups U$(m+n)$, S(f) and

Table 4. The ordinary versus the graded groups.

	U$(m+n)$, S(f), $\mathscr{S}(f)$	U(m/n), $\mathring{\text{S}}(f)$, $\mathring{\mathscr{S}}(f)$
Casimir invariants	$I_k^{m+n} = F_k(C(f), m+n)$	$I_k^{m/n} = F_k(\mathring{C}(f), m-n)$
Special Gel'fand basis	$\vert Y_m^\nu, W_k^\nu\rangle = P_m^{[\nu]k}\vert\omega\rangle$	$\vert \mathring{Y}_m^\nu, \mathring{W}_k^\nu\rangle = \mathring{P}_m^{[\nu]k}\vert\omega\rangle$
General Gel'fand basis	$\vert Y_m^\nu, W_\kappa^\nu\rangle = [R^{[\nu]k}(\omega)]^{-1}\vert Y_m^\nu W_\kappa^\nu\rangle_{\text{ass}}$	$\vert \mathring{Y}_m^\nu, \mathring{W}_\kappa^\nu\rangle = [\mathring{R}^{[\nu]k}(\omega)]^{-1}\vert \mathring{Y}_m^\nu \mathring{W}_k^\nu\rangle_{\text{ass}}$

$\mathcal{S}(f)$ to the case of U(m/n), $\mathring{S}(f)$ and $\mathring{\mathcal{S}}(f)$. For example, if $I_k^{m+n} = F_k(C(f), m+n)$ is the formula for the Casimir invariant of U$(m+n)$ as a function of the CSCO-I of S(f) and the quantity $(m+n)$, then by replacing the CSCO-I of S(f) with that of $\mathring{S}(f)$, and the quantity $(m+n)$ with $(m-n)$, we immediately get the formula for the Casimir invariants $I_k^{m/n}$ of U(m/n).

Once the Gel'fand basis of SU(m/n) is identified with the quasi-standard basis of the graded state permutation group $\mathring{\mathcal{S}}(f)$, the other seemingly more formidable problems, such as the Clebsch–Gordan coefficients, isoscalar factors etc of the group SU(m/n) can be solved easily from the permutation group representation. The matrix elements of the infinitesimal operators in the Gel'fand basis, the Clebsch–Gordan coefficients of SU(m/n), the isoscalar factors for the group chains SU$(mp+nq/mq+np) \supset$ SU$(m/n) \times$ SU(p/q), SU$(m+p/n+q) \supset$ SU$(m/n) \times$ SU(p/q), and SU$(m/n) \supset$ SU$(m) \times$ SU(n), etc will be investigated in the forthcoming papers.

Note added in proof. The same conclusion is obtained by A B Balantekin 1982 *J. Math. Phys.* **23** 486.

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